

A Carleson Problem for the Boussinesq Operator

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Abstract In this paper, we show that the Boussinesq operator $\mathcal{B}_t f$ converges pointwise to its initial data $f \in H^s(\mathbb{R})$ as $t \rightarrow 0$ provided $s \geq \frac{1}{4}$ —more precisely—on one hand, by constructing a counterexample in \mathbb{R} we discover that the optimal convergence index $s_{c,1} = \frac{1}{4}$; on the other hand, we find that the Hausdorff dimension of the divergence set for $\mathcal{B}_t f$ is

$$\alpha_{1,\mathcal{B}}(s) = \begin{cases} 1 - 2s, & \text{as } \frac{1}{4} \leq s \leq \frac{1}{2}; \\ 1, & \text{as } 0 < s < \frac{1}{4}. \end{cases}$$

Moreover, a higher dimensional lift was also obtained for f being radial.

Keywords Carleson problem, Boussinesq operator, pointwise convergence, Hausdorff dimension, Sobolev space

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1 Introduction

1.1 Carleson Problem for Schrödinger Operator

For $(n, s) \in \mathbb{N} \times \mathbb{R}$ and the Schwartz space $S(\mathbb{R}^n)$, let

$$H^s(\mathbb{R}^n) := \left\{ f \in S(\mathbb{R}^n) : \|f\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty \right\}$$

be the s -order L^2 Sobolev space on \mathbb{R}^n . Formally, the Schrödinger operator acting on $f \in S(\mathbb{R}^n)$

$$\mathcal{S}_t f(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|^2} \hat{f}(\xi) d\xi \quad (1.1)$$

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solves the initial data problem of free Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_x u = 0; \\ u(x, 0) = f(x). \end{cases} \quad (1.2)$$

Due to a fundamental interest in mathematical and theoretical physics, in [8] Carleson proposed a problem to determine the optimal order $s_{c,n}$ such that

$$\lim_{t \rightarrow 0} \mathcal{S}_t f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n \quad (1.3)$$

holds for all $f \in H^s(\mathbb{R}^n)$ with $s \geq s_{c,n}$.

There are enumerable literatures devoted to this problem (see [4–6, 24, 28, 30, 33, 35, 39–41] and the references therein)—in particular— $s_{c,n} = \frac{n}{2(n+1)}$. However, it is perhaps appropriate to mention some important steps toward the formula on $s_{c,n}$.

▷ For $n = 1$, Carleson [8] proved (1.1) converges to its initial data with $s \geq \frac{1}{4}$ and Dahlberg–Kenig [10] gave counterexample to show that this convergence cannot be true for $s < \frac{1}{4}$.

▷ For $n \geq 2$, Bourgain [6] and Sjölin [36] formulated independently counterexamples for $s < \frac{n}{2(n+1)}$. Recently, a positive result has been established under $s > \frac{n}{2(n+1)}$ by Du–Guth–Li [14] for $n = 2$ and Du–Zhang [16] for $n \geq 3$.

Furthermore, in [32] Sjögren–Sjölin refined Carleson’s problem to determine the Hausdorff dimension of the divergence set:

$$\alpha_{n,S}(s) := \sup_{f \in H^s(\mathbb{R}^n)} \dim_H \left\{ x \in \mathbb{R}^n : \lim_{t \rightarrow 0} \mathcal{S}_t f(x) \neq f(x) \right\}. \quad (1.4)$$

By Sobolev embedding, we easily get

$$\alpha_{n,S}(s) = 0, \quad \forall s > \frac{n}{2},$$

thereby being led to consider the case $s \leq \frac{n}{2}$.

▷ Bourgain’s counterexample in [6] implies

$$\alpha_{n,S}(s) = n, \quad \forall s < \frac{n}{2(n+1)}.$$

▷ Lucà–Rogers in [26] proved

$$\alpha_{n,S}(s) = n \quad \text{as } s = \frac{n}{2(n+1)}.$$

▷ For $\frac{n}{4} \leq s \leq \frac{n}{2}$, we can combine the results in Žubrinčić [42] and Barceló et al. [2] to obtain $\alpha_{n,S}(s) = n - 2s$.

▷ Notice that if

$$n = 1 \quad \& \quad \frac{n}{2(n+1)} = \frac{n}{4} = \frac{1}{4},$$

then $\alpha_{1,S}(s) = 1 - 2s$. And yet, for

$$n \geq 2 \quad \& \quad \frac{n}{2(n+1)} < s < \frac{n}{4},$$

nobody knows the value of $\alpha_{n,S}(s)$; see also [15, 16, 25–27] for more information.

1.2 Carleson Problem for Boussinesq Operator

As a nonlinear variant of (1.1), the Boussinesq operator acting on $f \in S(\mathbb{R}^n)$ is defined by

$$u(x, t) := \mathcal{B}_t f(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|\sqrt{1+|\xi|^2}} \hat{f}(\xi) d\xi, \quad (1.5)$$

which occurs in a large number of physical situations, which has motivated their study in physics and mathematics. The name of this operator comes from the Boussinesq equation (cf.[7])

$$u_{tt} - u_{xx} \pm u_{xxxx} = (u^2)_{xx}, \quad \forall (t, x) \in \mathbb{R}^2$$

modelling the propagation of long waves on the surface of water with small amplitude. Our interest in this operator arises from the study of the Gross–Pitaevskii (G-P) equation

$$i\partial_t \psi + \Delta \psi = (|\psi|^2 - 1)\psi \text{ subject to } \psi : \mathbb{R}^{1+n} \rightarrow \mathbb{C} \text{ \& } \lim_{|x| \rightarrow \infty} \psi = 1.$$

The above nonzero boundary condition arises naturally in physical contexts such as Bose–Einstein condensates, superfluids and nonlinear optics, or in the hydrodynamic interpretation of NLS (cf. [17]). There are many literatures studying the existence and asymptotic behavior of a solution to the G-P equation. For the most recent progress on these topics we refer the readers to [3, 18–22]. Even though (1.5) is very close to (1.1), the constant boundary condition brings a remarkable effect on the space-time behaviour of a solution—this actually is one of the main motivations of this paper.

Quite surprisingly, upon letting $v = \psi - 1$, we find

$$i\partial_t v + \Delta v - 2\Re v = v^2 + 2|v|^2 + |v|^2 v,$$

thereby using the diagonal transform

$$v = v_1 + iv_2 \rightarrow u = u_1 + iu_2 := v_1 + iUv_2,$$

to get the following system for (u, v) :

$$\begin{cases} i\partial_t u - Hu = U(3v_1^2 + v_2^2 + |v|^2 v_1) + i(2v_1 v_2 + |v|^2 v_2); \\ U := \sqrt{-\Delta(2 - \Delta)}^{-1}; \\ H := \sqrt{-\Delta(2 - \Delta)}. \end{cases}$$

Accordingly, (1.5) solves the initial data problem of the induced Boussinesq equation

$$\begin{cases} i\partial_t u - Hu = 0; \\ u(x, 0) = f(x). \end{cases} \quad (1.6)$$

In this paper, we are motivated by §1.1 and similarity between (1.1) (solving (1.2)) and (1.5) (solving (1.6)) to consider:

I. The Carleson problem for $\mathcal{B}_t f(x)$: evaluating the optimal $s_{c,n}$ such that

$$\lim_{t \rightarrow 0} \mathcal{B}_t f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^n \quad (1.7)$$

holds for any $f \in H^s(\mathbb{R}^n)$ with $s \geq s_{c,n}$.

II. A refinement of the Carleson problem for $\mathcal{B}_t f(x)$: determining the Hausdorff dimension of the divergence set:

$$\alpha_{n,\mathcal{B}}(s) := \sup_{f \in H^s(\mathbb{R}^n)} \dim_H \left\{ x \in \mathbb{R}^n : \lim_{t \rightarrow 0} \mathcal{B}_t f(x) \neq f(x) \right\}. \quad (1.8)$$

In order to resolve this issue, for any $f \in H^{s>0}(\mathbb{R}^n)$ we make the following decomposition

$$f(x) = f_{<1}(x) + f_{\geq 1}(x) \quad \& \quad \widehat{f_{<1}}(\xi) = \hat{f}(\xi)\phi(\xi),$$

where ϕ is a bump function based on the origin-centered ball $B(0, 2)$ with radius 2 and satisfies $\phi|_{B(0,1)} = 1$. Thus

$$f_{<1} \in H^s(\mathbb{R}^n), \quad \forall s > \frac{n}{2},$$

which implies

$$\lim_{t \rightarrow 0} \mathcal{B}_t(f_{<1})(x) = f_{<1}(x).$$

Accordingly, (1.7) amounts to

$$\lim_{t \rightarrow 0} \mathcal{B}_t(f_{\geq 1})(x) = f_{\geq 1}(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

Upon noticing

$$|\xi| \sqrt{1 + |\xi|^2} \approx |\xi|^2, \quad \forall |\xi| \geq 1,$$

we may guess that the Boussinesq operator should behave like the Schrödinger operator. However, such a guessing is not easily confirmed. Here are two instances.

▷ [14–16] used the l^2 -decoupling and polynomial partitioning to set up the following inequality:

$$\left\| \sup_{0 < t < 1} |\mathcal{S}_t(f)(x)| \right\|_{L^q(B(0,1))} \lesssim R^{\beta(n,q)} \|f\|_{L^2(\mathbb{R}^n)},$$

under a suitable condition:

$$\begin{cases} 1 < q < \infty; \\ \beta(n, q) > 0; \\ \text{supp } \hat{f} \subseteq A_R := \{\xi \in \mathbb{R}^n : |\xi| \approx \text{a dyadic number } R\}. \end{cases}$$

For this purpose, a scaling argument is essential. And \mathcal{S}_t is scaling invariant. But \mathcal{B}_t does not have this symmetry which is the main difficulty.

▷ Cho–Ko [9] extended the convergence result on \mathcal{S}_t to some generalized dispersive operators excluding the Boussinesq operator.

Nevertheless, we can still get a first outcome for the one-dimensional case as described below.

Theorem 1.1 *The optimal order $s_{c,1} = \frac{1}{4}$ follows from two assertions as seen below.*

(i) *If $s \geq \frac{1}{4}$ and $f \in H^s(\mathbb{R})$, then*

$$\lim_{t \rightarrow 0} \mathcal{B}_t f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}. \quad (1.9)$$

(ii) *For $0 < s < \frac{1}{4}$, there exist two disjoint compact intervals $I, J \subset \mathbb{R}$ and a function $f_0 \in H^s(\mathbb{R})$ supported in I such that*

$$\lim_{t \rightarrow 0} \mathcal{B}_t f_0(x) = 0 \quad \forall x \in J \text{ fails.}$$

Remark 1.2 We obtain sharp result when $n = 1$ in Theorem 1.1. By Van der Corput's lemma, we have Lemma 2.2 which is the key to the proof of Theorem 1.1. The proof of Lemma 2.2 is complex which is the main difficulty.

In order to state the second result on \mathbb{R} , we are required to introduce three more concepts.

▷ If $\psi(r) := e^{-r^2}$, then

$$\mathcal{B}_t^N f(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} \psi\left(\frac{|\xi|}{N}\right) e^{ix \cdot \xi} e^{it|\xi|\sqrt{1+|\xi|^2}} \hat{f}(\xi) d\xi$$

is called the truncated operator.

▷ A positive Borel measure μ is $(0, n] \ni \alpha$ -dimensional provided

$$c_\alpha(\mu) := \sup_{(x,r) \in \mathbb{R}^n \times (0,\infty)} r^{-\alpha} \mu(B(x,r)) < \infty \quad \text{with} \quad B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}.$$

Moreover, $M^\alpha(\mathbb{B}^n)$ is the class of the α -dimensional probability measures supported in the unit ball $\mathbb{B}^n = B(0, 1)$.

▷ If $A \lesssim B$ stands for $A \leq cB$ for a constant $c > 0$ and $A \approx B$ means $A \lesssim B \lesssim A$, then

$$\bar{\alpha}_{n,\mathcal{B}}(s) := \inf \left\{ \alpha : \frac{\| \sup_{(k,N) \in \mathbb{N}^2} |\mathcal{B}_{t_k}^N f| \|_{L^1(d\mu)}}{\sqrt{c_\alpha(\mu)} \|f\|_{H^s(\mathbb{R}^n)}} \lesssim 1, \right. \\ \left. \forall (\mu, f, t_k) \in M^\alpha(\mathbb{B}^n) \times H^s(\mathbb{R}^n) \times (0, \infty) \right\}. \quad (1.10)$$

Usually, we will define $\bar{\alpha}_{n,\mathcal{B}}(s) = +\infty$ if the set is empty. From [2] and an equivalence of the Hausdorff capacity in [1, Section 3] it follows that

$$\dim_H \left\{ x \in \mathbb{R}^n : \lim_{k \rightarrow \infty} \mathcal{B}_{t_k}^N f(x) \neq f(x) \right\} \leq \bar{\alpha}_{n,\mathcal{B}}(s) \quad \text{as } f \in H^s(\mathbb{R}^n) \text{ \& } t_k \rightarrow 0. \quad (1.11)$$

Theorem 1.3 *Let*

$$\begin{cases} \frac{1}{4} \leq s \leq \frac{1}{2}; \\ \alpha > 1 - 2s; \\ \mu \in M^\alpha(\mathbb{B}); \\ \lim_{k \rightarrow \infty} t_k = 0. \end{cases}$$

Then:

(i)

$$\left\| \sup_{(k,N) \in \mathbb{N}^2} |\mathcal{B}_{t_k}^N f| \right\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)} \|f\|_{H^s(\mathbb{R})}; \quad (1.12)$$

(ii) $\bar{\alpha}_{1,\mathcal{B}}(s) = 1 - 2s = \alpha_{1,\mathcal{B}}(s)$.

Remark 1.4 Similar problems were studied by Ding–Niu [11], Ding–Niu [12] and Lan–Li–Niu [23]. Moreover, the Boussinesq operator was studied in [11]. In which the authors set up the pointwise convergence of Boussinesq operator when $s > \frac{1}{2}$. In [12] the authors considered a class of dispersive operators with some growth condition (H1), (H2) and (H3). They set up the pointwise convergence under the condition $s \geq 1/4$. For the same class of dispersive operators, Theorem 1.3 was also proved in [23]. Boussinesq operator we study in this paper does not verify the growth conditions (H1)–(H3). But as we pointed out in the above, the low frequency part does not effect the convergence property, which means one can also remove the condition (H1) in [12] and [23]. With this observation, we can conclude that their results can also be extend to the class of dispersive operators including Boussinesq operator. Another contribution of this paper is the counterexamples in Theorem 1.1. With the counterexample we can conclude that the critical index $s_c = \frac{1}{4}$ and $\alpha_{x,\mathcal{B}} = 1$ for $0 < s < \frac{1}{4}$.

While extending Theorems 1.1–1.3 to a higher dimension, we are suggested by [31] (handling the Carleson problem for certain generalized dispersive equation) to consider the radial case, thereby discovering the following assertion.

Theorem 1.5 *For*

$$\left\{ \begin{array}{l} n \geq 2; \\ \frac{1}{4} \leq s < \frac{1}{2}; \\ 2 \leq q \leq \frac{2}{1-2s}; \\ x \in \mathbb{R}^n; \\ f \in H^s(\mathbb{R}^n), \end{array} \right.$$

let

$$\mathcal{B}^{**}f(x) := \sup_{t \in \mathbb{R}} |\mathcal{B}_t f(x)|.$$

Then

$$\left(\int_{\mathbb{R}^n} |\mathcal{B}^{**}f(x)|^q |x|^\alpha dx \right)^{\frac{1}{q}} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^n)}, \quad \forall \text{ radial } f \in H^s(\mathbb{R}^n) \quad (1.13)$$

if and only if

$$\alpha = q \left(\frac{n}{2} - s \right) - n.$$

Consequently, under this situation we have

$$\lim_{t \rightarrow 0} \mathcal{B}_t f(x) = f(x), \quad a.e. \ x \in \mathbb{R}^n. \quad (1.14)$$

Remark 1.6 Ding–Niu [13] obtained (1.14) by setting up the L^q estimates for a class of maximal operators including $\mathcal{B}^{**}f(x)$. In this paper, we set up the corresponding weighted L^q boundedness as (1.13). Such an argument was motivated by Niu’s works [31] for a class of dispersive operators which does not include Boussinesq operator. Our another contribution here is to find the sharp power α in (1.13). The main tools include the linearization of the maximal operator, TT^* argument and Bessel’s function. Moreover, Lemma 2.2 in Section 2 plays a crucial role in the proof of Theorem 1.5.

The rest of this paper is designed below to present a much-more-involved proof of the above three theorems according to a twofold argument style.

2 Demonstration of Theorem 1.1

2.1 Proof of Theorem 1.1 (i)

First of all, recall the following well-known variant of van der Corput’s lemma.

Lemma 2.1 ([37, pp. 332–334]) *For $a < b$ and $I = [a, b]$, let $\Phi \in C^\infty(I)$ be real-valued and $\psi \in C^\infty(I)$.*

(i) *If $|\Phi'(x)| \geq \gamma > 0, \forall x \in I$ and Φ' is monotonic on I , then*

$$\left| \int_I e^{i\Phi(x)} \psi(x) dx \right| \lesssim \frac{1}{\gamma} \left(|\psi(b)| + \int_I |\psi'(x)| dx \right).$$

(ii) If $|\Phi''(x)| \geq \gamma > 0, \forall x \in I$, then

$$\left| \int_I e^{i\Phi(x)} \psi(x) dx \right| \lesssim \frac{1}{\gamma^{\frac{1}{2}}} \left(|\psi(b)| + \int_I |\psi'(x)| dx \right).$$

Next, we need a crucial oscillatory estimate whose fractional order Schrödinger operator analogue was considered in Sjölin [34].

Lemma 2.2 *Let*

$$\begin{cases} \frac{1}{2} \leq s < 1; \\ 0 < t < \frac{1}{6}; \\ q \in C_0^\infty(\mathbb{R}). \end{cases}$$

Then

$$\left| \int_{\mathbb{R}} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q\left(\frac{\xi}{N}\right) d\xi \right| \lesssim \frac{1}{|x|^{1-s}}$$

for $x \in \mathbb{R}$ and $N > 1$.

Proof of Lemma 2.2 Without loss of generality, we may assume $x \neq 0$.

$$\begin{aligned} \int_{\mathbb{R}} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q\left(\frac{\xi}{N}\right) d\xi &= \int_{|\xi| < |x|^{-1}} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q\left(\frac{\xi}{N}\right) d\xi \\ &\quad + \int_{|\xi| \geq |x|^{-1}} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q\left(\frac{\xi}{N}\right) d\xi \\ &=: A + B. \end{aligned}$$

It is easy to estimate

$$|A| \lesssim \int_{|\xi| < |x|^{-1}} |\xi|^{-s} d\xi \approx \frac{1}{|x|^{1-s}}.$$

However, estimating B is divided into two cases.

Case 1 $|x| < 1$. Two subcases are considered below.

(i) Under $|x|^2 \leq \frac{t}{2}$, we set

$$\Phi(\xi) := x \cdot \xi + t\xi\sqrt{1+\xi^2}, \quad \forall \xi \geq |x|^{-1}.$$

Then

$$\Phi'(\xi) = x + t \frac{1+2\xi^2}{\sqrt{1+\xi^2}} = x \left(1 + \frac{t}{x} \frac{1+2\xi^2}{\sqrt{1+\xi^2}} \right).$$

Note that

$$\left| 1 + \frac{t}{x} \frac{1+2\xi^2}{\sqrt{1+\xi^2}} \right| \geq \left| \frac{t}{x} \frac{1+2\xi^2}{\sqrt{1+\xi^2}} \right| - 1 \geq \frac{2|x|^2}{|x|} \frac{2\xi^2}{\sqrt{2\xi}} - 1 \geq 2\sqrt{2}|x|\xi - 1 \geq 1.$$

So, $|\Phi'(\xi)| \geq |x|$.

Also, we have

$$\Phi''(\xi) = t \frac{3\xi + 2\xi^3}{(1+\xi^2)^{\frac{3}{2}}},$$

thereby getting that Φ' is monotonic for $\xi \geq |x|^{-1}$. Upon setting

$$\psi(\xi) := \xi^{-s} q\left(\frac{\xi}{N}\right),$$

we obtain

$$\begin{aligned}
 |\psi(\xi)| &\lesssim |x|^{-s}, \quad \forall x \geq |x|^{-1}. \\
 \int_{|x|^{-1}}^{\infty} |\psi'(\xi)| d\xi &= \int_{|x|^{-1}}^{\infty} \left| \xi^{-s} \frac{1}{N} q' \left(\frac{\xi}{N} \right) - s \xi^{-s-1} q \left(\frac{\xi}{N} \right) \right| d\xi \\
 &\lesssim |x|^s \int_{|x|^{-1}}^{\infty} \frac{1}{N} \left| q' \left(\frac{\xi}{N} \right) \right| d\xi + \int_{|x|^{-1}}^{\infty} \xi^{-s-1} d\xi \\
 &\lesssim |x|^s.
 \end{aligned}$$

By Lemma 2.1, we then conclude that

$$\begin{cases} \left| \int_{|x|^{-1}}^{\infty} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q \left(\frac{\xi}{N} \right) d\xi \right| \lesssim \frac{1}{|x|} |x|^s \approx \frac{1}{|x|^{1-s}}; \\ \left| \int_{-\infty}^{-|x|^{-1}} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q \left(\frac{\xi}{N} \right) d\xi \right| \lesssim \frac{1}{|x|} |x|^s \approx \frac{1}{|x|^{1-s}}. \end{cases}$$

(ii) Under $|x|^2 > \frac{t}{2}$, we achieve

$$\int_{|x|^{-1}}^{\infty} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q \left(\frac{\xi}{N} \right) d\xi =: B_1 + B_2 + B_3,$$

where

$$\begin{cases} B_1 := \int_{I_1} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q \left(\frac{\xi}{N} \right) d\xi; \\ B_2 := \int_{I_2} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q \left(\frac{\xi}{N} \right) d\xi; \\ B_3 := \int_{I_3} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q \left(\frac{\xi}{N} \right) d\xi, \end{cases}$$

and

$$\begin{cases} I_1 := \left\{ \xi \geq |x|^{-1}, \xi \leq \delta \frac{|x|}{t} \right\}; \\ I_2 := \left\{ \xi \geq |x|^{-1}, \delta \frac{|x|}{t} \leq \xi \leq K \frac{|x|}{t} \right\}; \\ I_3 := \left\{ \xi \geq |x|^{-1}, \xi \geq K \frac{|x|}{t} \right\}; \\ \delta > 0 \text{ \& } K > 0 \text{ are small \& large numbers respectively.} \end{cases}$$

For $\xi \in I_1$ we have

$$t \frac{1+2\xi^2}{\sqrt{1+\xi^2}} \leq t \frac{3\xi^2}{\xi} \leq 3t\delta \frac{|x|}{t} \leq \frac{1}{2}|x|,$$

whence

$$|\Phi'(\xi)| \geq |x| - t \frac{1+2\xi^2}{\sqrt{1+\xi^2}} \geq |x| - \frac{1}{2}|x| = \frac{1}{2}|x|.$$

By Lemma 2.1, we can obtain

$$|B_1| \lesssim \frac{1}{|x|^{1-s}}.$$

Meanwhile, upon estimating

$$t \frac{1+2\xi^2}{\sqrt{1+\xi^2}} \geq t \frac{2\xi^2}{\sqrt{2}\xi} \geq \sqrt{2}tK \frac{|x|}{t} \geq 2|x|, \quad \forall \xi \in I_3,$$

we get

$$|\Phi'(\xi)| \geq t \frac{1+2\xi^2}{\sqrt{1+\xi^2}} - |x| \geq |x|, \quad \forall \xi \in I_3.$$

By Lemma 2.1, we have

$$|B_3| \lesssim \frac{1}{|x|^{1-s}}.$$

To estimate I_2 , note that

$$\begin{cases} \Phi''(\xi) = t \frac{3\xi + 2\xi^3}{(1+\xi^2)^{\frac{3}{2}}} \geq t \frac{2\xi^3}{(2\xi^2)^{\frac{3}{2}}} = \frac{\sqrt{2}}{2}t, & \forall \xi \in I_2; \\ |\psi(\xi)| = \left| \xi^{-s} q\left(\frac{\xi}{N}\right) \right| \lesssim \left(\delta \frac{|x|}{t} \right)^{-s} \lesssim \left(\frac{|x|}{t} \right)^{-s}, \end{cases}$$

and

$$\begin{aligned} \int_{I_2} |\psi'(\xi)| d\xi &= \int_{I_2} \left| \xi^{-s} \frac{1}{N} q'\left(\frac{\xi}{N}\right) - s \xi^{-s-1} q\left(\frac{\xi}{N}\right) \right| d\xi \\ &\leq \int_{I_2} |\xi|^{-s} \frac{1}{N} \left| q'\left(\frac{\xi}{N}\right) \right| d\xi + C \int_{I_2} \xi^{-s-1} d\xi \\ &\leq \left(\delta \frac{|x|}{t} \right)^{-s} \int_{I_2} \frac{1}{N} \left| q'\left(\frac{\xi}{N}\right) \right| d\xi + C \left(\frac{|x|}{t} \right)^{-s} \\ &\lesssim \left(\frac{|x|}{t} \right)^{-s}. \end{aligned}$$

So, by Lemma 2.1 and $s \geq \frac{1}{2}$, we obtain

$$|B_2| \lesssim t^{-\frac{1}{2}} \left(\frac{|x|}{t} \right)^{-s} \approx t^{s-\frac{1}{2}} |x|^{-s} \lesssim |x|^{2(s-\frac{1}{2})} |x|^{-s} \approx \frac{1}{|x|^{1-s}},$$

thereby finding

$$\begin{cases} \left| \int_{|x|^{-1}}^{\infty} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q\left(\frac{\xi}{N}\right) d\xi \right| \lesssim \frac{1}{|x|^{1-s}}; \\ \left| \int_{-\infty}^{-|x|^{-1}} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q\left(\frac{\xi}{N}\right) d\xi \right| \lesssim \frac{1}{|x|^{1-s}}. \end{cases}$$

Case 2 $|x| > 1$. Upon writing

$$\begin{aligned} \int_{|x|^{-1}}^{\infty} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q\left(\frac{\xi}{N}\right) d\xi &:= \int_{|x|^{-1}}^1 e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q\left(\frac{\xi}{N}\right) d\xi \\ &\quad + \int_1^{\infty} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q\left(\frac{\xi}{N}\right) d\xi \\ &=: B_4 + B_5, \end{aligned}$$

we utilize the technique similar to handling **Case 1** to gain

$$|B_5| \lesssim \frac{1}{|x|^{1-s}}.$$

Concerning B_4 , we estimate

$$\begin{cases} |\Phi'(\xi)| = \left| x + t \frac{1+2\xi^2}{\sqrt{1+\xi^2}} \right| \geq |x| \left(1 - \left| \frac{t}{x} \frac{1+2\xi^2}{\sqrt{1+\xi^2}} \right| \right) \geq \frac{|x|}{2}, & \forall 0 < t < \frac{1}{6}; \\ |\psi(\xi)| = \left| \xi^{-s} q\left(\frac{\xi}{N}\right) \right| \lesssim \xi^{-s} \lesssim |x|^s, & \forall \xi > |x|^{-1}, \end{cases}$$

and

$$\begin{aligned}
\int_{|x|^{-1}}^1 |\psi'(\xi)| d\xi &= \int_{|x|^{-1}}^1 \left| \xi^{-s} \frac{1}{N} q' \left(\frac{\xi}{N} \right) - s \xi^{-s-1} q \left(\frac{\xi}{N} \right) \right| d\xi \\
&\leq \int_{|x|^{-1}}^1 \xi^{-s} \frac{1}{N} \left| q' \left(\frac{\xi}{N} \right) \right| d\xi + C \int_{|x|^{-1}}^1 \xi^{-s-1} d\xi \\
&\lesssim |x|^s \int_{|x|^{-1}}^1 \frac{1}{N} \left| q' \left(\frac{\xi}{N} \right) \right| d\xi + |x|^s \\
&\lesssim |x|^s.
\end{aligned}$$

By Lemma 2.1, we have

$$|B_4| = \left| \int_{|x|^{-1}}^1 e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q \left(\frac{\xi}{N} \right) d\xi \right| \lesssim \frac{1}{|x|^{1-s}},$$

thereby reaching

$$\begin{cases} \left| \int_{|x|^{-1}}^\infty e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q \left(\frac{\xi}{N} \right) d\xi \right| \lesssim \frac{1}{|x|^{1-s}}; \\ \left| \int_{-\infty}^{-|x|^{-1}} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} |\xi|^{-s} q \left(\frac{\xi}{N} \right) d\xi \right| \lesssim \frac{1}{|x|^{1-s}}. \end{cases}$$

Finally, we arrive at

Proof of Theorem 1.1 (i) It is enough to prove that the following estimate

$$\left(\int_B |\mathcal{B}^* f(x)|^2 dx \right)^{\frac{1}{2}} \lesssim \|f\|_{\dot{H}^s(\mathbb{R})}, \quad f \in \dot{H}^s(\mathbb{R}) \quad (2.1)$$

holds for all balls B in \mathbb{R} due to the fact that (2.1) implies Theorem 1.1 (i) and

$$\mathcal{B}^* f(x) := \sup_{0 < t < 1} |\mathcal{B}_t f(x)|.$$

To do so, set

$$Rf(x) := \phi(x) \int_{\mathbb{R}} e^{ix \cdot \xi} e^{it(x)|\xi|\sqrt{1+\xi^2}} |\xi|^{-s} \hat{f}(\xi) d\xi, \quad x \in \mathbb{R}, f \in S(\mathbb{R}),$$

where $\phi(x)$ is a real-valued function in $C_c^\infty(\mathbb{R})$, $t(x)$ is a measurable function of x with $0 < t(x) < 1$ and $s = \frac{1}{4}$. It suffices to prove that the operator R is bounded on $L^2(\mathbb{R})$.

Upon letting

$$p(x, \xi) := \phi(x) e^{it(x)|\xi|\sqrt{1+\xi^2}} |\xi|^{-s},$$

we get

$$Rf(x) = \int_{\mathbb{R}} e^{ix \cdot \xi} p(x, \xi) \hat{f}(\xi) d\xi.$$

Via choosing a real-valued function $\rho \in C_c^\infty(\mathbb{R})$ such that

$$\begin{cases} \rho(\xi) = \begin{cases} 1 & \text{as } |\xi| \leq 1; \\ 0 & \text{as } |\xi| \geq 2; \end{cases} \\ \rho_N(\xi) := \rho \left(\frac{\xi}{N} \right), \quad \forall N \in \mathbb{N}, \end{cases}$$

and defining

$$\begin{cases} p_N(x, \xi) := \rho_N(\xi)p(x, \xi); \\ R_N f(x) := \int_{\mathbb{R}} e^{ix \cdot \xi} p_N(x, \xi) \hat{f}(\xi) d\xi, \quad f \in S(\mathbb{R}). \end{cases}$$

we can readily see that R_N is bounded on $L^2(\mathbb{R})$ and its adjoint operator is given by

$$R_N^* h(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(x-y) \cdot \xi} \overline{p_N(y, \xi)} h(y) dy d\xi, \quad h \in S(\mathbb{R}).$$

According to Plancherel's theorem, we have

$$\begin{aligned} & \int_{\mathbb{R}} |R_N^* h(x)|^2 dx \\ &= \int_{\mathbb{R}} |\widehat{R_N^* h}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}} \widehat{R_N^* h}(\xi) \overline{\widehat{R_N^* h}(\xi)} d\xi \\ &= (2\pi)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-iy \cdot \xi} \overline{p_N(y, \xi)} h(y) dy \int_{\mathbb{R}} e^{-iz \cdot \xi} \overline{p_N(z, \xi)} h(z) dz d\xi \\ &= (2\pi)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-iy \cdot \xi} \overline{p_N(y, \xi)} h(y) dy \int_{\mathbb{R}} e^{iz \cdot \xi} p_N(z, \xi) \overline{h(z)} dz d\xi \\ &= (2\pi)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i[(z-y) \cdot \xi + (t(z)-t(y))|\xi|\sqrt{1+\xi^2}]} \rho_N^2(\xi) |\xi|^{-2s} d\xi \phi(y) \phi(z) h(y) \overline{h(z)} dy dz \\ &= (2\pi)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} K_N(y, z) h(y) \overline{h(z)} dy dz, \end{aligned}$$

where

$$K_N(y, z) := \phi(y) \phi(z) \int_{\mathbb{R}} e^{i[(z-y) \cdot \xi + (t(z)-t(y))|\xi|\sqrt{1+\xi^2}]} \rho_N^2(\xi) |\xi|^{-2s} d\xi \quad \& \quad s = \frac{1}{4}.$$

By Lemma 2.2, we have

$$|K_N(y, z)| \lesssim |\phi(y)| |\phi(z)| |y - z|^{-\frac{1}{2}}, \quad N > 1, \quad (2.2)$$

thereby getting

$$\begin{aligned} \int_{\mathbb{R}} |R_N^* h(x)|^2 dx &= (2\pi)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} K_N(y, z) h(y) \overline{h(z)} dy dz \\ &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |y - z|^{-\frac{1}{2}} |\phi(y) \phi(z)| |h(y) h(z)| dy dz \\ &\approx \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\phi(z) h(z)|}{|y - z|^{\frac{1}{2}}} dz |\phi(y) h(y)| dy \\ &\approx \int_{\mathbb{R}} I_{\frac{1}{2}}(|\phi h|)(y) |\phi(y) h(y)| dy \\ &\lesssim \|I_{\frac{1}{2}}(|\phi h|)\|_{L^4(\mathbb{R})} \|\phi h\|_{L^{\frac{4}{3}}(\mathbb{R})} \\ &\lesssim \|\phi h\|_{L^{\frac{4}{3}}(\mathbb{R})}^2 \\ &\lesssim \|\phi\|_{L^4(\mathbb{R})}^2 \|h\|_{L^2(\mathbb{R})}^2 \\ &\lesssim \|h\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

where $I_{\frac{1}{2}}$ denotes the Riesz potential operator of order $\frac{1}{2}$ which is bounded from $L^{\frac{4}{3}}(\mathbb{R})$ to $L^4(\mathbb{R})$, and Hölder's inequality has been utilized two times. Accordingly, R_N is uniformly bounded on $L^2(\mathbb{R})$. Since

$$Rf(x) = \lim_{N \rightarrow \infty} R_N f(x), \quad f \in S(\mathbb{R}),$$

we use Fatou's lemma to get

$$\int_{\mathbb{R}} |Rf(x)|^2 dx \leq \liminf_{N \rightarrow \infty} \int_{\mathbb{R}} |R_N f(x)|^2 dx \lesssim \|f\|_{L^2(\mathbb{R})}^2,$$

thereby establishing $L^2(\mathbb{R})$ -boundedness of R . □

2.2 Proof of Theorem 1.1 (ii)

To verify Theorem 1.1 (ii), we need four lemmas.

First, we are about to use the fact that the dispersive waves with different frequencies transport at different velocities—more clearly—we take

$$\begin{cases} g \in S(\mathbb{R}); \\ \text{supp } \check{g} \subset (-1, 1); \\ \check{g}(0) \neq 0; \\ f_v(x) := e^{-\frac{ix}{v^2}} \check{g}\left(\frac{x}{v}\right), \quad 0 < v < 1. \end{cases}$$

Note that

$$\text{supp } f_v \subset (-v, v) \quad \& \quad f_v \in S(\mathbb{R}).$$

So, we have

Lemma 2.3 ([10]) *If*

$$0 < s < \frac{1}{4} \quad \& \quad 0 < v < 1,$$

then

$$\widehat{f}_v(\xi) = vg\left(v\xi + \frac{1}{v}\right) \quad \& \quad \|f_v\|_{H^s(\mathbb{R})} \lesssim v^{\frac{1}{2}-2s}.$$

Second, this last lemma, along with Heisenberg's inequality, leads to

Lemma 2.4 *For*

$$\Phi(\xi) := |\xi| \sqrt{1 + \xi^2},$$

there exists a triple $\{c_0 > 0, \delta > 0, v_0 > 0\}$ such that

$$|\mathcal{B}_t f_v(x)| \geq c_0$$

holds for any triple $\{v, x, t\}$ with

$$\begin{cases} 0 < v < v_0; \\ 0 < x < \delta; \\ t = \frac{x}{\Phi'(\frac{1}{v^2})}. \end{cases}$$

Proof First, since

$$\check{g}(0) \neq 0 \Rightarrow \int_{\mathbb{R}} g(\xi) d\xi \neq 0,$$

we choose a large number L such that

$$\int_{|\xi| \geq L} |g(\xi)| d\xi \leq \frac{1}{100} \left| \int_{\mathbb{R}} g(\xi) d\xi \right|,$$

thereby evaluating

$$\begin{aligned} \mathcal{B}_t f_v(x) &= \int_{\mathbb{R}} e^{i(x \cdot \xi + t\Phi(\xi))} \widehat{f}_v(\xi) d\xi \\ &= \int_{\mathbb{R}} e^{i(x \cdot \xi + t\Phi(\xi))} v g\left(v\xi + \frac{1}{v}\right) d\xi \\ &= \int_{\mathbb{R}} e^{i\left(\frac{x}{v}(\xi - \frac{1}{v}) + t\Phi(\frac{1}{v}|\xi - \frac{1}{v}|)\right)} g(\xi) d\xi \\ &=: \int_{\mathbb{R}} e^{iF} g(\xi) d\xi, \end{aligned}$$

where

$$F(x, t, \xi, v) := \frac{x}{v} \left(\xi - \frac{1}{v} \right) + t\Phi \left(\frac{1}{v} \left| \xi - \frac{1}{v} \right| \right).$$

Via choosing $v_0 := \frac{1}{2L}$, we get that if $0 < v < v_0$ then

$$\begin{aligned} |\mathcal{B}_t f_v(x)| &\geq \left| \int_{|\xi| \leq L} e^{iF} g(\xi) d\xi \right| - \left| \int_{|\xi| \geq L} e^{iF} g(\xi) d\xi \right| \\ &\geq \left| \int_{|\xi| \leq L} e^{iF} g(\xi) d\xi \right| - \int_{|\xi| \geq L} |g(\xi)| d\xi \\ &\geq \left| \int_{|\xi| \leq L} e^{iF} g(\xi) d\xi \right| - \frac{1}{100} \left| \int_{\mathbb{R}} g(\xi) d\xi \right|. \end{aligned}$$

Next we consider

$$\left| \int_{|\xi| \leq L} e^{iF} g(\xi) d\xi \right|.$$

With the help of Taylor's expansion we have

$$\begin{aligned} \Phi \left(\frac{1}{v} \left| \xi - \frac{1}{v} \right| \right) &= \Phi \left(\frac{1}{v^2} - \frac{\xi}{v} \right) \\ &= \Phi \left(\frac{1}{v^2} \right) - \frac{\xi}{v} \Phi' \left(\frac{1}{v^2} \right) + \frac{\xi^2}{v^2} \frac{\Phi''(\frac{1}{v^2})}{2} + O \left(-\frac{\xi}{v} \right)^3, \end{aligned}$$

whence

$$F(x, t, \xi, v) = \frac{x \cdot \xi}{v} - \frac{x}{v^2} + t\Phi \left(\frac{1}{v^2} \right) - t\frac{\xi}{v} \Phi' \left(\frac{1}{v^2} \right) + t\frac{\xi^2}{v^2} \frac{\Phi''(\frac{1}{v^2})}{2} + O \left[t \left(-\frac{\xi}{v} \right)^3 \right].$$

An application of

$$t := \frac{x}{\Phi'(\frac{1}{v^2})}$$

yields

$$F(x, t, \xi, v) = -\frac{x}{v^2} + \frac{x\Phi(\frac{1}{v^2})}{\Phi'(\frac{1}{v^2})} + \frac{x}{\Phi'(\frac{1}{v^2})} \frac{\Phi''(\frac{1}{v^2})}{2} \frac{\xi^2}{v^2} + O \left(-\frac{x}{\Phi'(\frac{1}{v^2})} \frac{\xi^3}{v^3} \right).$$

Therefore, for small $\delta > 0$ we have

$$\begin{aligned}
& \left| \int_{|\xi| \leq L} e^{iF} g(\xi) d\xi \right| \\
&= \left| \int_{-L}^L e^{i\left(\frac{x}{\Phi'(\frac{1}{v^2})} - \frac{\Phi''(\frac{1}{v^2})}{2} \frac{\xi^2}{v^2}\right)} e^{iO\left(-\frac{x}{\Phi'(\frac{1}{v^2})} \frac{\xi^3}{v^3}\right)} g(\xi) d\xi \right| \\
&= \left| \int_{-L}^L e^{i\left(\frac{x}{\Phi'(\frac{1}{v^2})} - \frac{\Phi''(\frac{1}{v^2})}{2} \frac{\xi^2}{v^2}\right)} g(\xi) d\xi + \int_{-L}^L e^{i\left(\frac{x}{\Phi'(\frac{1}{v^2})} - \frac{\Phi''(\frac{1}{v^2})}{2} \frac{\xi^2}{v^2}\right)} \left[e^{iO\left(-\frac{x}{\Phi'(\frac{1}{v^2})} \frac{\xi^3}{v^3}\right)} - 1 \right] g(\xi) d\xi \right| \\
&\gtrsim \left| \int_{-L}^L e^{i\left(\frac{x}{\Phi'(\frac{1}{v^2})} - \frac{\Phi''(\frac{1}{v^2})}{2} \frac{\xi^2}{v^2}\right)} g(\xi) d\xi \right| - x \\
&\geq \frac{1}{2} \left| \int_{-L}^L g(\xi) d\xi \right|, \quad \forall x \in (0, \delta).
\end{aligned}$$

Finally, if

$$0 < v < v_0 \quad \& \quad 0 < x < \delta,$$

then

$$\begin{aligned}
|\mathcal{B}_t f_v(x)| &\geq \frac{1}{2} \left| \int_{-L}^L g(\xi) d\xi \right| - \frac{1}{100} \left| \int_{\mathbb{R}} g(\xi) d\xi \right| \\
&\geq \frac{1}{2} \left| \int_{\mathbb{R}} g(\xi) d\xi \right| - \frac{1}{200} \left| \int_{\mathbb{R}} g(\xi) d\xi \right| - \frac{1}{100} \left| \int_{\mathbb{R}} g(\xi) d\xi \right| \\
&\geq \frac{1}{4} \left| \int_{\mathbb{R}} g(\xi) d\xi \right|,
\end{aligned}$$

as desired. \square

Third, although the following lemma is known as the dispersion estimate, we will use it to check the wave with high frequency spread so fast such that in the test interval $\frac{\delta}{2} < x < \delta$, the wave is already gone at the given time.

Lemma 2.5 *If*

$$\begin{cases} 0 < v < \min \left\{ v_0, \frac{\delta}{4} \right\}; \\ 0 < t < 1; \\ \frac{\delta}{2} < x < \delta, \end{cases}$$

then

$$|\mathcal{B}_t f_v(x)| \lesssim \frac{v}{t^{\frac{1}{2}}}.$$

Proof Let

$$\begin{cases} m(\xi) := e^{it|\xi|\sqrt{1+\xi^2}}; \\ \hat{m}(y) := \int_{\mathbb{R}} e^{it(|\xi|\sqrt{1+\xi^2} - \frac{y \cdot \xi}{t})} d\xi. \end{cases}$$

Then two situations are handled below.

\triangleright Under $|\xi| \leq 1$, it is easy to see that

$$|\hat{m}(y)| \lesssim 1 \lesssim t^{-\frac{1}{2}}.$$

▷ Under $|\xi| > 1$, we set

$$\Psi(\xi) := |\xi|\sqrt{1+\xi^2} - \frac{y \cdot \xi}{t}.$$

If $\xi \geq 0$, then

$$\Psi'(\xi) = \frac{1+2\xi^2}{(1+\xi^2)^{\frac{3}{2}}} - \frac{y}{t} \quad \& \quad \Psi''(\xi) = \frac{3\xi+2\xi^3}{(1+\xi^2)^{\frac{3}{2}}}.$$

Upon letting

$$\Psi'(\xi_0) = 0 = \frac{1+2\xi_0^2}{(1+\xi_0^2)^{\frac{3}{2}}} - \frac{y}{t},$$

we get

$$\begin{cases} t = \frac{y}{\Phi'(\xi_0)} = \frac{x}{\Phi'(\frac{1}{v^2})}; \\ \xi_0 \approx \frac{1}{v^2}; \\ |\Psi''(\xi_0)| \approx \left| \Psi''\left(\frac{1}{v^2}\right) \right| \gtrsim 1. \end{cases}$$

According to stationary phase method again, we can obtain

$$|\hat{m}(y)| \lesssim \frac{1}{\sqrt{t\Psi''(\xi_0)}} \lesssim t^{-\frac{1}{2}},$$

whence

$$\begin{aligned} |\mathcal{B}_t f_v(x)| &= \left| \int_{\mathbb{R}} e^{i(x \cdot \xi + t\Phi(\xi))} \hat{f}_v(\xi) d\xi \right| \\ &= \left| \int_{\mathbb{R}} \hat{m}(y) f_v(x-y) dy \right| \\ &\leq \int_{\mathbb{R}} |\hat{m}(y)| |f_v(x-y)| dy \\ &\lesssim \int_{\mathbb{R}} \frac{1}{t^{\frac{1}{2}}} |f_v(y)| dy \\ &\approx \frac{1}{t^{\frac{1}{2}}} \int_{\mathbb{R}} \left| e^{-i\frac{y^2}{v^2}} \check{g}\left(\frac{y}{v}\right) \right| dy \\ &\approx \frac{1}{t^{\frac{1}{2}}} v \int_{\mathbb{R}} |\check{g}(s)| ds \\ &\lesssim \frac{v}{t^{\frac{1}{2}}}. \end{aligned} \quad \square$$

Fourth, the coming-up-next lemma will be used to check the wave with low frequency which cannot arrive at the test interval $\frac{\delta}{2} < x < \delta$ at given time.

Lemma 2.6 *If*

$$\begin{cases} 0 < v < \min\left\{v_0, \frac{\delta}{4}\right\}; \\ 0 < t < 1; \\ \frac{\delta}{2} < x < \delta, \end{cases}$$

then

$$|\mathcal{B}_t f_v(x)| \lesssim \frac{t}{v^4}.$$

Proof According to the definition of $f_v(x)$, we have

$$\begin{aligned}\mathcal{B}_t f_v(x) &= \int_{\mathbb{R}} e^{ix \cdot \xi} e^{it|\xi|\sqrt{1+\xi^2}} \widehat{f}_v(\xi) d\xi \\ &= \int_{\mathbb{R}} (e^{it|\xi|\sqrt{1+\xi^2}} - 1) e^{ix \cdot \xi} \widehat{f}_v(\xi) d\xi + 2\pi f_v(x),\end{aligned}$$

thereby getting $f_v(x) = 0$ due to

$$\begin{cases} \text{supp } f_v \subset (-v, v); \\ 0 < v < \frac{\delta}{4}; \\ \frac{\delta}{2} < x < \delta. \end{cases}$$

Consequently,

$$\begin{aligned}|\mathcal{B}_t f_v(x)| &= \left| \int_{\mathbb{R}} (e^{it|\xi|\sqrt{1+\xi^2}} - 1) e^{ix \cdot \xi} \widehat{f}_v(\xi) d\xi \right| \\ &\lesssim \int_{|\xi| \leq 1} t|\xi|\sqrt{1+\xi^2} |\widehat{f}_v(\xi)| d\xi + \int_{|\xi| > 1} t|\xi|\sqrt{1+\xi^2} |\widehat{f}_v(\xi)| d\xi \\ &=: I_1 + I_2.\end{aligned}$$

On one hand, we have

$$\begin{aligned}I_1 &\lesssim \int_{|\xi| \leq 1} t|\xi| \left| v\xi + \frac{1}{v} \right| d\xi \\ &\lesssim \frac{t}{v} \int_{\mathbb{R}} \left| \eta - \frac{1}{v} \right| |g(\eta)| d\eta \\ &\lesssim \frac{t}{v} \int_{\mathbb{R}} \left(|\eta| + \frac{1}{v} \right) |g(\eta)| d\eta \\ &\lesssim \frac{t}{v^2}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}I_2 &\lesssim \int_{|\xi| > 1} t|\xi|^2 \left| v\xi + \frac{1}{v} \right| d\xi \\ &\lesssim \frac{t}{v^2} \int_{\mathbb{R}} \left| \eta - \frac{1}{v} \right|^2 |g(\eta)| d\eta \\ &\lesssim \frac{t}{v^2} \int_{\mathbb{R}} \left(|\eta|^2 + \frac{1}{v^2} \right) |g(\eta)| d\eta \\ &\lesssim \frac{t}{v^4}.\end{aligned}$$

Combining the estimate of I_1 with I_2 , we have

$$|\mathcal{B}_t f_v(x)| \lesssim \frac{t}{v^4}.$$

□

With the aid of the previous four lemmas, we come to

Proof of Theorem 1.1 (ii) Let

$$\begin{cases} 0 < v_1 < \min \left\{ v_0, \frac{\delta}{4} \right\}; \\ \varepsilon_k := 2^{-k} \quad \text{for } k = 1, 2, 3, \dots; \\ v_k := \varepsilon_k v_{k-1}^2 \quad \text{for } k = 2, 3, 4, \dots \end{cases}$$

By induction, we have

$$\begin{cases} 0 < v_k < 1 \quad \text{for } k = 1, 2, 3, \dots; \\ 0 < v_k \leq \varepsilon_k \quad \text{for } k = 1, 2, 3, \dots; \\ v_k \leq \varepsilon_k v_{k-1} \leq \frac{1}{2} v_{k-1} \quad \text{for } k = 2, 3, 4, \dots; \\ \sum_{j=k+1}^{\infty} v_j \lesssim v_{k+1}; \\ \sum_{j=1}^{k-1} \frac{1}{v_j^4} \lesssim \frac{1}{v_{k-1}^4}. \end{cases}$$

Upon defining

$$f := \sum_{k=1}^{\infty} f_{v_k}.$$

and using Lemma 2.3, we obtain

$$\|f\|_{H^s(\mathbb{R})} \leq \sum_{k=1}^{\infty} \|f_{v_k}\|_{H^s(\mathbb{R})} \lesssim \sum_{k=1}^{\infty} v_k^{\frac{1}{2}-2s} \leq \sum_{k=1}^{\infty} 2^{-k(\frac{1}{2}-2s)} < \infty, \quad \forall 0 < s < \frac{1}{4}.$$

Also, via utilizing

$$\text{supp } f \subseteq \left(-\frac{\delta}{4}, \frac{\delta}{4} \right)$$

and $t(x)$ in Lemma 2.4:

$$t(x) = \frac{x}{\Phi'(\frac{1}{v^2})} = \frac{xv^2\sqrt{v^4+1}}{v^4+2},$$

we achieve

$$\begin{aligned} |\mathcal{B}_{t_k(x)} f(x)| &= \left| \sum_{j=1}^{\infty} \mathcal{B}_{t_k(x)} f_{v_j}(x) \right| \\ &\geq |\mathcal{B}_{t_k(x)} f_{v_k}(x)| - \left| \sum_{j=1}^{k-1} \mathcal{B}_{t_k(x)} f_{v_j}(x) \right| - \left| \sum_{j=k+1}^{\infty} \mathcal{B}_{t_k(x)} f_{v_j}(x) \right| \\ &\geq c_0 - \left| \sum_{j=1}^{k-1} \mathcal{B}_{t_k(x)} f_{v_j}(x) \right| - \left| \sum_{j=k+1}^{\infty} \mathcal{B}_{t_k(x)} f_{v_j}(x) \right|, \end{aligned}$$

and then estimate the last two sums under $\frac{\delta}{2} < x < \delta$.

▷ For $1 \leq j \leq k-1$, by Lemma 2.6, we get

$$\left| \sum_{j=1}^{k-1} \mathcal{B}_{t_k(x)} f_{v_j}(x) \right| \lesssim \sum_{j=1}^{k-1} \frac{t_k(x)}{v_j^4}$$

$$\begin{aligned}
&\approx \frac{xv_k^2\sqrt{v_k^4+1}}{v_k^4+2} \frac{1}{v_{k-1}^4} \\
&\lesssim \frac{(\varepsilon_k v_{k-1}^2)^2 \sqrt{(\varepsilon_k v_{k-1}^2)^4+1}}{(\varepsilon_k v_{k-1}^2)^4+2} \frac{1}{v_{k-1}^4} \\
&\rightarrow 0 \quad \text{as } k \rightarrow \infty.
\end{aligned}$$

▷ For $j \geq k+1$, by Lemma 2.5, we have

$$\begin{aligned}
\left| \sum_{j=k+1}^{\infty} \mathcal{B}_{t_k(x)} f_{v_j}(x) \right| &\lesssim \sum_{j=k+1}^{\infty} \frac{v_j}{t_k^{\frac{1}{2}}(x)} \\
&\lesssim \frac{1}{t_k^{\frac{1}{2}}(x)} v_{k+1} \\
&\lesssim \left(\frac{v_k^4+2}{v_k^2\sqrt{v_k^4+1}} \right)^{\frac{1}{2}} \varepsilon_{k+1} v_k^2 \\
&\approx \frac{v_k\sqrt{v_k^4+2}}{\sqrt[4]{v_k^4+1}} \varepsilon_{k+1} \\
&\rightarrow 0 \quad \text{as } k \rightarrow \infty.
\end{aligned}$$

Accordingly, for $k \geq k_0$, we can get

$$|\mathcal{B}_{t_k(x)} f(x)| \geq \frac{c_0}{2} \quad \text{for } \frac{\delta}{2} < x < \delta. \quad \square$$

3 Demonstration of Theorem 1.3

3.1 Proof of Theorem 1.3 (i)

This amounts to verifying (1.12).

Proof of Theorem 1.3 (i) It is enough to deal with $\frac{1}{4} \leq s < \frac{1}{2}$ since the case $s = \frac{1}{2}$ follows as a consequence. The α -energy of μ is defined by

$$I_\alpha(\mu) := \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{1}{|x-y|^\alpha} d\mu(x) d\mu(y).$$

From a dyadic decomposition it follows that if

$$\mu \in M^\alpha(\mathbb{B}) \quad \& \quad \alpha > 1-2s$$

then

$$\begin{aligned}
I_{1-2s}(\mu) &= \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{1}{|x-y|^{1-2s}} d\mu(x) d\mu(y) \\
&= \int_{\mathbb{B}} \sum_{j=0}^{\infty} \int_{B(y, 2^{-j}) \setminus B(y, 2^{-j-1})} \frac{1}{|x-y|^{1-2s}} d\mu(x) d\mu(y) \\
&\leq \int_{\mathbb{B}} \sum_{j=0}^{\infty} \int_{B(y, 2^{-j})} \frac{1}{2^{(-j-1)(1-2s)}} d\mu(x) d\mu(y) \\
&\leq \int_{\mathbb{B}} \sum_{j=0}^{\infty} 2^{(j+1)(1-2s)} \mu(B(y, 2^{-j})) d\mu(y)
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{B}} \sum_{j=0}^{\infty} 2^{(j+1)(1-2s)} c_{\alpha}(\mu) 2^{-j\alpha} d\mu(y) \\
&\leq \int_{\mathbb{B}} c_{\alpha}(\mu) 2^{1-2s} \sum_{j=0}^{\infty} 2^{j(1-2s-\alpha)} d\mu(y) \\
&\lesssim c_{\alpha}(\mu).
\end{aligned}$$

Accordingly, it suffices to prove that

$$\left\| \sup_{(k,N) \in \mathbb{N}^2} |\mathcal{B}_{t_k}^N f| \right\|_{L^1(d\mu)} \lesssim \sqrt{I_{1-2s}(\mu)} \|f\|_{\dot{H}^s(\mathbb{R})}, \quad (3.1)$$

equivalently,

$$\left| \int_{\mathbb{B}} \mathcal{B}_{t(x)}^{N(x)} f(x) \omega(x) d\mu(x) \right|^2 \lesssim I_{1-2s}(\mu) \|f\|_{\dot{H}^s(\mathbb{R})}^2, \quad (3.2)$$

holds uniformly in the measurable functions

$$\begin{cases} t(x) : \mathbb{B} \rightarrow \mathbb{R}; \\ N(x) : \mathbb{B} \rightarrow [1, \infty); \\ \omega(x) : \mathbb{B} \rightarrow \mathbb{S}^1 = \{-1, 1\}. \end{cases}$$

By applying Fubini's theorem and Hölder's inequality we can get

$$\begin{aligned}
&\left| \int_{\mathbb{B}} \mathcal{B}_{t(x)}^{N(x)} f(x) \omega(x) d\mu(x) \right|^2 \\
&\leq \int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\xi|^{2s} d\xi \int_{\mathbb{B}} \left| \int_{\mathbb{B}} \psi\left(\frac{|\xi|}{N(x)}\right) e^{i(x \cdot \xi + t(x)|\xi|\sqrt{1+|\xi|^2})} \omega(x) d\mu(x) \right|^2 \frac{1}{|\xi|^{2s}} d\xi \\
&= \|f\|_{\dot{H}^s(\mathbb{R})}^2 \int_{\mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{R}} \psi\left(\frac{|\xi|}{N(x)}\right) \psi\left(\frac{|\xi|}{N(y)}\right) e^{i[(x-y) \cdot \xi - (t(x)-t(y))|\xi|\sqrt{1+|\xi|^2}]} \frac{1}{|\xi|^{2s}} d\xi \\
&\quad \times \omega(x) \omega(y) d\mu(x) d\mu(y).
\end{aligned}$$

In order to prove (3.2) it suffices to show that

$$\begin{aligned}
&\int \int \int_{\mathbb{B} \times \mathbb{B} \times \mathbb{R}} (\cdots) \\
&:= \int_{\mathbb{B}} \int_{\mathbb{B}} \int_{\mathbb{R}} \psi\left(\frac{|\xi|}{N(x)}\right) \psi\left(\frac{|\xi|}{N(y)}\right) e^{i[(x-y) \cdot \xi - (t(x)-t(y))|\xi|\sqrt{1+|\xi|^2}]} \frac{1}{|\xi|^{2s}} d\xi \omega(x) \omega(y) d\mu(x) d\mu(y) \\
&\lesssim I_{1-2s}(\mu)
\end{aligned} \quad (3.3)$$

holds uniformly in the above-defined functions t, N, ω . By Lemma 2.2, we get

$$\begin{aligned}
&\int \int \int_{\mathbb{B} \times \mathbb{B} \times \mathbb{R}} (\cdots) \lesssim \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{1}{|x-y|^{1-2s}} |\omega(x) \omega(y)| d\mu(x) d\mu(y) \\
&\approx \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{1}{|x-y|^{1-2s}} d\mu(x) d\mu(y) \\
&\approx I_{1-2s}(\mu),
\end{aligned}$$

thereby reaching (3.3). □

3.2 Proof of Theorem 1.3 (ii)

This consists of two-side-inequalities.

Proof From (1.12) and the definition of $\bar{\alpha}_{1,\mathcal{B}}(s)$, we have

$$\bar{\alpha}_{1,\mathcal{B}}(s) \leq 1 - 2s.$$

Next we show

$$\bar{\alpha}_{1,\mathcal{B}}(s) \geq 1 - 2s.$$

To do so, consider

$$\begin{cases} \hat{f} = \chi_A; \\ d\mu(x) = N^n \chi_E(x) dx; \\ A = B(0, N); \\ E = B(0, N^{-1}), \end{cases}$$

Clearly, we have

$$\mathcal{B}_t^N f(x) = (2\pi)^{-n} \int_{B(0,N)} \psi\left(\frac{|\xi|}{N}\right) e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} d\xi.$$

▷ On one hand, upon choosing $t = N^{-2}$, we see that the phase is close to zero for all $x \in E$, thereby deriving

$$\begin{aligned} \left\| \sup_{0 < t < 1} |\mathcal{B}_t^N f(x)| \right\|_{L^1(d\mu)} &= \left\| \sup_{0 < t < 1} \left| (2\pi)^{-n} \int_{B(0,N)} \psi\left(\frac{|\xi|}{N}\right) e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} d\xi \right| \right\|_{L^1(d\mu)} \\ &\gtrsim \left\| \int_{B(0,N)} \psi\left(\frac{|\xi|}{N}\right) d\xi \right\|_{L^1(d\mu)} \\ &\approx \left\| \int_{B(0,N)} \psi\left(\frac{|\xi|}{N}\right) d\xi \right\|_{L^1(N^n \chi_E(x) dx)} \\ &\gtrsim N^n |A| |E| \\ &\approx N^n. \end{aligned}$$

▷ On the other hand, we calculate

$$\begin{aligned} \sqrt{c_\alpha(\mu)} \|f\|_{H^s(\mathbb{R}^n)} &= \sqrt{\sup_{(x,r) \in \mathbb{R}^n \times (0,\infty)} r^{-\alpha} \mu(B(x,r))} \left(\int_{B(0,N)} (1 + |\xi|^2)^s d\xi \right)^{\frac{1}{2}} \\ &\leq \sqrt{\sup_{(x,r) \in \mathbb{R}^n \times (0,\infty)} r^{-\alpha} N^n |B(0, N^{-1}) \cap B(x,r)| (1 + N^2)^{\frac{s}{2}} N^{\frac{n}{2}}} \\ &\lesssim \sqrt{\frac{N^n N^{-n}}{N^{-\alpha}}} N^{s + \frac{n}{2}} \\ &\lesssim N^{\frac{\alpha}{2}} N^{s + \frac{n}{2}}. \end{aligned}$$

Via letting $N \rightarrow \infty$, we get

$$\bar{\alpha}_{n,\mathcal{B}}(s) \geq n - 2s,$$

thereby taking $n = 1$ to reveal

$$\bar{\alpha}_{1,\mathcal{B}}(s) \geq 1 - 2s. \quad \square$$

4 Demonstration of Theorem 1.5

4.1 Proof of Theorem 1.5 (if)

In order to prove the if-part, we recall two lemmas.

Lemma 4.1 ([38], Bessel's formula) *If $J_m(r)$ is the Bessel function defined by*

$$J_m(r) = \frac{\left(\frac{r}{2}\right)^m}{\Gamma(m + \frac{1}{2})\pi^{\frac{1}{2}}} \int_{-1}^1 e^{irx} (1-x^2)^{m-\frac{1}{2}} dx \quad \text{as } m > -\frac{1}{2},$$

then

$$J_m(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi m}{2} - \frac{\pi}{4}\right) + O(r^{-\frac{3}{2}}) \quad \text{as } r \rightarrow \infty.$$

Lemma 4.2 ([29], Pitt's inequality) *If*

$$\begin{cases} r \geq p; \\ 0 \leq \alpha < \frac{1}{r}; \\ 0 \leq \alpha_1 < 1 - \frac{1}{p}; \\ \alpha_1 = \alpha + 1 - \frac{1}{r} - \frac{1}{p}, \end{cases}$$

then

$$\left(\int_{\mathbb{R}} |\hat{f}(\xi)|^r |\xi|^{-\alpha r} d\xi \right)^{\frac{1}{r}} \lesssim \left(\int_{\mathbb{R}} |f(x)|^p |x|^{\alpha_1 p} dx \right)^{\frac{1}{p}}.$$

Proof of Theorem 1.5 (if) It suffices to prove the if-part with $s = \frac{1}{4}$ and f being radial.

First, [38] gives

$$\hat{f}(\xi) = (2\pi)^{\frac{n}{2}} |\xi|^{1-\frac{n}{2}} \int_0^\infty f(r) J_{\frac{n}{2}-1}(r|\xi|) r^{\frac{n}{2}} dr.$$

Second, we set $t(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ be radial measurable and

$$\mathcal{B}_{t(x)} f(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi + it(x)|\xi| \sqrt{1+|\xi|^2}} \hat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n,$$

whence

$$\mathcal{B}_{t(u)} f(u) = (2\pi)^{\frac{n}{2}-n} u^{1-\frac{n}{2}} \int_0^\infty J_{\frac{n}{2}-1}(ru) e^{it(u)r\sqrt{1+r^2}} \hat{f}(r) r^{\frac{n}{2}} dr, \quad u > 0,$$

where

$$\begin{cases} \mathcal{B}_{t(u)} f(u) = \mathcal{B}_{t(x)} f(x) & \text{as } u = |x|; \\ \hat{f}(r) = \hat{f}(\xi) & \text{as } r = |\xi|. \end{cases}$$

Third, in order to obtain (1.13), it remains to prove

$$\left(\int_0^\infty |\mathcal{B}_{t(u)} f(u)|^q u^{q(\frac{n}{2}-s)-1} du \right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty |\hat{f}(r)|^2 r^{2s} r^{n-1} dr \right)^{\frac{1}{2}}. \quad (4.1)$$

The following three steps will be carried out.

▷ Let

$$\begin{cases} g(r) := \hat{f}(r) r^s r^{\frac{n}{2}-\frac{1}{2}} \quad \forall r > 0; \\ \left(\int_0^\infty |\hat{f}(r)|^2 r^{2s} r^{n-1} dr \right)^{\frac{1}{2}} = \left(\int_0^\infty |g(r)|^2 dr \right)^{\frac{1}{2}}. \end{cases}$$

▷ In order to control the left hand side of (4.1), we estimate

$$\begin{aligned}\mathcal{B}_{t(u)}f(u)u^{\frac{n}{2}-s-\frac{1}{q}} &= (2\pi)^{\frac{n}{2}-n}u^{1-\frac{n}{2}}u^{\frac{n}{2}-s-\frac{1}{q}}\int_0^\infty J_{\frac{n}{2}-1}(ru)e^{it(u)r\sqrt{1+r^2}}\hat{f}(r)r^{\frac{n}{2}}dr \\ &= (2\pi)^{-\frac{n}{2}}u^{1-s-\frac{1}{q}}\int_0^\infty J_{\frac{n}{2}-1}(ru)e^{it(u)r\sqrt{1+r^2}}r^{-s}r^{\frac{1}{2}}g(r)dr \\ &=: (2\pi)^{-\frac{n}{2}}Dg(u),\end{aligned}$$

where

$$Dg(u) := u^{1-s-\frac{1}{q}}\int_0^\infty J_{\frac{n}{2}-1}(ru)e^{it(u)r\sqrt{1+r^2}}r^{-s}r^{\frac{1}{2}}g(r)dr, \quad r > 0.$$

and consequently,

$$\left(\int_0^\infty |\mathcal{B}_{t(u)}f(u)|^q u^{q(\frac{n}{2}-s)-1} du\right)^{\frac{1}{q}} = \left(\int_0^\infty |(2\pi)^{-\frac{n}{2}}Dg(u)|^q du\right)^{\frac{1}{q}}.$$

It reduces to prove

$$\left(\int_0^\infty |Dg(u)|^q du\right)^{\frac{1}{q}} \lesssim \left(\int_0^\infty |g(r)|^2 dr\right)^{\frac{1}{2}}.$$

But, since the adjoint operator of D is given by

$$D^*h(r) = r^{-s}r^{\frac{1}{2}}\int_0^\infty J_{\frac{n}{2}-1}(ru)e^{-it(u)r\sqrt{1+r^2}}u^{1-s-\frac{1}{q}}h(u)du, \quad r > 0,$$

it suffices to prove

$$\|D^*h\|_{L^2(0,\infty)} \lesssim \|h\|_{L^p(0,\infty)},$$

where

$$\begin{cases} \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{4} \leq s < \frac{1}{2}; \\ 2 \leq q \leq \frac{2}{1-2s}; \\ 1 < \frac{2}{1+2s} \leq p \leq 2. \end{cases}$$

Upon setting

$$\sigma := \frac{1}{q} + s - \frac{1}{2},$$

we see

$$D^*h(r) = r^{-s}\int_0^\infty (ru)^{\frac{1}{2}}J_{\frac{n}{2}-1}(ru)e^{-it(u)r\sqrt{1+r^2}}u^{-\sigma}h(u)du.$$

Therefore, we are led to estimate the Bessel function in the last formula.

• By Lemma 4.1, there exist b_1 and b_2 depending only on n such that

$$t^{\frac{1}{2}}J_{\frac{n}{2}-1}(t) = b_1e^{it} + b_2e^{-it} + O\left(\min\left\{1, \frac{1}{t}\right\}\right), \quad t > 0. \quad (4.2)$$

In fact, Lemma 4.1 yields

$$J_{\frac{n}{2}-1}(t) = \sqrt{\frac{2}{\pi t}}\cos\left(t - \frac{\pi(n-1)}{4}\right) + O(t^{-\frac{3}{2}}) \quad \text{as } t \rightarrow \infty,$$

and so

$$\begin{aligned}
t^{\frac{1}{2}} J_{\frac{n}{2}-1}(t) &= t^{\frac{1}{2}} \sqrt{\frac{2}{\pi t}} \cos\left(t - \frac{\pi(n-1)}{4}\right) + O(t^{-1}) \\
&= \sqrt{\frac{2}{\pi}} \cos\left(\frac{\pi(n-1)}{4}\right) \cos t + \sqrt{\frac{2}{\pi}} \sin\left(\frac{\pi(n-1)}{4}\right) \sin t + O(t^{-1}) \\
&= (b_1 + b_2) \cos t - i(b_1 - b_2) \sin t + O(t^{-1}) \\
&= b_1 e^{it} + b_2 e^{-it} + O(t^{-1}),
\end{aligned}$$

where

$$\begin{aligned}
b_1 &= \frac{1}{2} \sqrt{\frac{2}{\pi}} \left(\cos\left(\frac{\pi(n-1)}{4}\right) - i \sin\left(\frac{\pi(n-1)}{4}\right) \right), \\
b_2 &= \frac{1}{2} \sqrt{\frac{2}{\pi}} \left(\cos\left(\frac{\pi(n-1)}{4}\right) + i \sin\left(\frac{\pi(n-1)}{4}\right) \right).
\end{aligned}$$

- When $t > 1$, we have

$$|t^{\frac{1}{2}} J_{\frac{n}{2}-1}(t) - (b_1 e^{it} + b_2 e^{-it})| \lesssim t^{-1}. \quad (4.3)$$

- For the case of $0 < t < 1$ and $n \geq 2$, we have

$$\begin{cases} J_m(t) = \frac{(\frac{t}{2})^m}{\Gamma(m + \frac{1}{2})\pi^{\frac{1}{2}}} \int_{-1}^1 e^{itx} (1-x^2)^{m-\frac{1}{2}} dx & \text{as } m > -\frac{1}{2}; \\ |J_m(t)| \lesssim t^m & \text{as } m > -\frac{1}{2} \text{ \& } t > 0; \\ |J_m(t)| \lesssim t^{-\frac{1}{2}} & \text{as } m > -\frac{1}{2} \text{ \& } 0 < t \leq 1; \\ |J_{\frac{n}{2}-1}(t)| \lesssim t^{-\frac{1}{2}} & \text{as } 0 < t \leq 1. \end{cases}$$

Accordingly, if $0 < t \leq 1$, then

$$\begin{aligned}
|t^{\frac{1}{2}} J_{\frac{n}{2}-1}(t) - (b_1 e^{it} + b_2 e^{-it})| &\leq |t^{\frac{1}{2}} J_{\frac{n}{2}-1}(t)| + |b_1 e^{it}| + |b_2 e^{-it}| \\
&\lesssim t^{\frac{1}{2}} t^{-\frac{1}{2}} + |b_1| + |b_2| \\
&\lesssim 1,
\end{aligned}$$

and hence

$$|t^{\frac{1}{2}} J_{\frac{n}{2}-1}(t) - (b_1 e^{it} + b_2 e^{-it})| \lesssim 1. \quad (4.4)$$

▷ Upon combining (4.3) with (4.4), we get (4.2). Now, (4.2) derives

$$\begin{cases} |t^{\frac{1}{2}} J_{\frac{n}{2}-1}(t) - (b_1 e^{it} + b_2 e^{-it})| \lesssim \frac{1}{t}, & t > 1; \\ |t^{\frac{1}{2}} J_{\frac{n}{2}-1}(t) - (b_1 e^{it} + b_2 e^{-it})| \lesssim 1, & 0 < t \leq 1, \end{cases} \quad (4.5)$$

where b_1 and b_2 depend only on n . With the help of (4.5) we obtain

$$D^* h(r) := b_1 B_1(r) + b_2 B_2(r) + B_3(r), \quad (4.6)$$

where

$$\begin{cases} B_1(r) = r^{-s} \int_0^\infty e^{iru} e^{-it(u)r\sqrt{1+r^2}} u^{-\sigma} h(u) du; \\ B_2(r) = r^{-s} \int_0^\infty e^{-iru} e^{-it(u)r\sqrt{1+r^2}} u^{-\sigma} h(u) du; \\ |B_3(r)| \leq Cr^{-s} \int_0^\infty \min\left\{1, \frac{1}{ru}\right\} u^{-\sigma} |h(u)| du. \end{cases}$$

In what follows, we estimate $B_1(r)$, $B_2(r)$ and $B_3(r)$ respectively.

- Via defining

$$B(r) := r^{-s} \int_0^\infty e^{iru} e^{-it(u)|r|\sqrt{1+|r|^2}} u^{-\sigma} h(u) du, \quad r \in \mathbb{R},$$

and choosing a real-valued function $\rho \in C_c^\infty(\mathbb{R})$ such that

$$\begin{cases} \rho(\xi) = \begin{cases} 1 & \text{as } |\xi| \leq 1; \\ 0 & \text{as } |\xi| \geq 2; \end{cases} \\ \rho_N(\xi) := \rho\left(\frac{\xi}{N}\right), \quad \forall N \in \mathbb{N}, \end{cases}$$

as well as letting

$$B_N(r) := \rho_N(r) |r|^{-s} \int_0^\infty e^{iru} e^{-it(u)|r|\sqrt{1+|r|^2}} u^{-\sigma} h(u) du,$$

we utilize Fubini's theorem to get

$$\int_{\mathbb{R}} |B_N(r)|^2 dr := \int_0^\infty \int_0^\infty I(u, v) u^{-\sigma} h(u) v^{-\sigma} \overline{h(v)} du dv, \quad (4.7)$$

where

$$I(u, v) := \int_{\mathbb{R}} e^{i((u-v)r - (t(u)-t(v))|r|\sqrt{1+|r|^2})} |r|^{-s} \rho_N^2(r) dr.$$

- By Lemma 2.2, we have

$$|I(u, v)| \lesssim \frac{1}{|u - v|^{\frac{1}{2}}} \quad \text{under } s = \frac{1}{4}. \quad (4.8)$$

From (4.7), (4.8) and Parseval's equality, we have

$$\begin{aligned} \|B_N\|_{L^2(\mathbb{R})}^2 &\lesssim \int_0^\infty \int_0^\infty \frac{1}{|u - v|^{\frac{1}{2}}} u^{-\sigma} h(u) v^{-\sigma} |h(v)| du dv \\ &\approx \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{|u - v|^{\frac{1}{2}}} u^{-\sigma} h_1(u) v^{-\sigma} |h_1(v)| du dv \\ &\approx \int_{\mathbb{R}} I_{\frac{1}{2}}(v^{-\sigma} |h_1(v)|)(u) u^{-\sigma} |h_1(u)| du \\ &\approx \int_{\mathbb{R}} |\xi|^{-\frac{1}{2}} (u^{-\sigma} |h_1(u)|)^\wedge(\xi) \overline{(u^{-\sigma} |h_1(u)|)^\wedge(\xi)} d\xi \\ &\approx \int_{\mathbb{R}} |\xi|^{-\frac{1}{2}} |(u^{-\sigma} |h_1(u)|)^\wedge(\xi)|^2 d\xi, \end{aligned}$$

where

$$h_1(u) = \begin{cases} h(u) & \text{as } u \geq 0; \\ 0 & \text{as } u < 0. \end{cases}$$

- By Pitt's inequality, we have

$$\left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\xi|^{-2s} d\xi \right)^{\frac{1}{2}} \lesssim \left(\int_{\mathbb{R}} |f(x)|^p |x|^{(s+\frac{1}{2})p-1} dx \right)^{\frac{1}{p}},$$

where

$$\begin{cases} \frac{1}{4} \leq s < \frac{1}{2}; \\ \frac{2}{1+2s} \leq p \leq 2. \end{cases}$$

Consequently,

$$\begin{aligned}
\|B_N\|_{L^2(\mathbb{R})}^2 &\lesssim \left(\int_{\mathbb{R}} |u^{-\sigma} h_1(u)|^p |u|^{\frac{3}{4}p-1} du \right)^{\frac{2}{p}} \\
&\approx \left(\int_{\mathbb{R}} u^{-\sigma p + \frac{3}{4}p-1} |h_1(u)|^p du \right)^{\frac{2}{p}} \\
&\approx \|h_1\|_{L^p(\mathbb{R})}^2 \\
&\approx \|h\|_{L^p(0,\infty)}^2,
\end{aligned}$$

namely,

$$\|B_N\|_{L^2(\mathbb{R})} \lesssim \|h\|_{L^p(0,\infty)}.$$

According to Fatou's lemma we obtain

$$\left(\int_{\mathbb{R}} |B(r)|^2 dr \right)^{\frac{1}{2}} \lesssim \|h\|_{L^p(0,\infty)}.$$

Furthermore,

$$\left(\int_0^\infty |B_i(r)|^2 dr \right)^{\frac{1}{2}} \lesssim \left(\int_{\mathbb{R}} |B(r)|^2 dr \right)^{\frac{1}{2}} \lesssim \|h\|_{L^p(0,\infty)}, \quad \forall i = 1, 2.$$

- Next we dominate $B_3(r)$ according to two situations.

(i) Under $0 < r < 1$, it suffices to prove

$$\left(\int_0^1 |B_3(r)|^2 dr \right)^{\frac{1}{2}} \lesssim \|h\|_{L^p(0,\infty)}.$$

Hölder's inequality derives

$$\begin{aligned}
|B_3(r)| &\lesssim \int_0^\infty \min \left\{ 1, \frac{1}{ru} \right\} u^{-\sigma} |h(u)| du \\
&\lesssim \int_0^{\frac{1}{r}} u^{-\sigma} |h(u)| du + \frac{1}{r} \int_{\frac{1}{r}}^\infty u^{-1-\sigma} |h(u)| du \\
&\lesssim \left(\int_0^{\frac{1}{r}} u^{-\sigma q} du \right)^{\frac{1}{q}} \|h\|_{L^p(0,\infty)} + \frac{1}{r} \left(\int_{\frac{1}{r}}^\infty u^{(-1-\sigma)q} du \right)^{\frac{1}{q}} \|h\|_{L^p(0,\infty)} \\
&\approx r^{\sigma - \frac{1}{q}} \|h\|_{L^p(0,\infty)} \\
&\approx r^{s - \frac{1}{2}} \|h\|_{L^p(0,\infty)},
\end{aligned}$$

whence

$$\left(\int_0^1 |B_3(r)|^2 dr \right)^{\frac{1}{2}} \lesssim \left(\int_0^1 r^{2s-1} dr \right)^{\frac{1}{2}} \|h\|_{L^p(0,\infty)} \lesssim \|h\|_{L^p(0,\infty)}.$$

(ii) Under $r \geq 1$, it suffices to prove

$$\left(\int_1^\infty |B_3(r)|^2 dr \right)^{\frac{1}{2}} \lesssim \|h\|_{L^p(0,\infty)}.$$

On one hand, since

$$|B_3(r)| \lesssim r^{-s} \int_0^{\frac{1}{r}} u^{-\sigma} |h(u)| du + r^{-s} \int_{\frac{1}{r}}^\infty \frac{1}{ru} u^{-\sigma} |h(u)| du$$

$$\begin{aligned}
&\approx r^{-s} \int_0^{\frac{1}{r}} u^{-\sigma} |h(u)| du + r^{-1-s} \int_{\frac{1}{r}}^{\infty} u^{-\sigma-1} |h(u)| du \\
&=: E_1(r) + E_2(r),
\end{aligned}$$

setting

$$F_1(t) := \frac{1}{t} E_1\left(\frac{1}{t}\right), \quad 0 < t < 1$$

yields

$$\begin{aligned}
F_1(t) &= t^{-1+s} \int_0^t u^{-\sigma} |h(u)| du \\
&\leq \int_0^t (t-u)^{-1+s} u^{-\sigma} |h(u)| du \\
&\leq \int_{\mathbb{R}} |t-u|^{-1+s} u^{-\sigma} |h_1(u)| du \\
&\leq I_s(u^{-\sigma} |h_1(u)|)(t),
\end{aligned}$$

Similarly to the estimate for $\|B_N\|_{L^2(\mathbb{R})}$, we have

$$\begin{aligned}
\left(\int_1^{\infty} |E_1(r)|^2 dr \right)^{\frac{1}{2}} &\lesssim \left(\int_0^1 |F_1(t)|^2 dt \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{\mathbb{R}} |I_s(u^{-\sigma} |h_1(u)|)(t)|^2 dt \right)^{\frac{1}{2}} \\
&\approx \left(\int_{\mathbb{R}} |(u^{-\sigma} |h_1(u)|)^{\wedge}(\xi)|^2 |\xi|^{-2s} d\xi \right)^{\frac{1}{2}} \\
&\lesssim \|h_1\|_{L^p(\mathbb{R})} \\
&\approx \|h\|_{L^p(0,\infty)},
\end{aligned}$$

thereby producing

$$\left(\int_1^{\infty} |E_1(r)|^2 dr \right)^{\frac{1}{2}} \lesssim \|h\|_{L^p(0,\infty)}.$$

On the other hand, we want to prove

$$\left(\int_1^{\infty} |E_2(r)|^2 dr \right)^{\frac{1}{2}} \lesssim \|h\|_{L^p(0,\infty)}.$$

Now, setting

$$F_2(t) := \frac{1}{t} E_2\left(\frac{1}{t}\right), \quad 0 < t < 1$$

implies

$$\begin{aligned}
F_2(t) &= t^s \int_t^{\infty} u^{-1-\sigma} |h(u)| du \\
&\leq \int_t^{\infty} u^s u^{-1-\sigma} |h(u)| du \\
&\leq \int_{\mathbb{R}} |t-u|^{-1+s} u^{-\sigma} |h_1(u)| du \\
&\leq I_s(u^{-\sigma} |h_1(u)|)(t).
\end{aligned}$$

Similarly to the estimate of

$$\left(\int_1^\infty |E_1(r)|^2 dr \right)^{\frac{1}{2}},$$

we have

$$\left(\int_1^\infty |E_2(r)|^2 dr \right)^{\frac{1}{2}} \lesssim \|h\|_{L^p(0,\infty)}.$$

Combining (4.6) with the estimates for $B_1(r)-B_2(r)-B_3(r)$ yields (4.1) and then (1.13). \square

4.2 Proof of Theorem 1.5 (only-if)

This part is constructive.

Proof of Theorem 1.5 (only-if) To do so, choose a nonnegative radial function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} \text{supp } \varphi \subset \{\xi : 1 \leq |\xi| \leq 2\}; \\ \varphi(\xi) = 1 \quad \text{as } \frac{5}{4} \leq |\xi| \leq \frac{7}{4}; \\ \hat{f}(\xi) := \varphi\left(\frac{\xi}{\lambda}\right) \quad \text{as } \lambda > 0. \end{cases}$$

It is easy to see that

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} \approx \lambda^{\frac{n}{2}+s}.$$

Since

$$\begin{aligned} \mathcal{B}_t f(x) &= \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|\sqrt{1+|\xi|^2}} \varphi\left(\frac{\xi}{\lambda}\right) d\xi \\ &= \lambda^n \int_{\mathbb{R}^n} e^{i\lambda x \cdot \eta} e^{it|\lambda \eta|\sqrt{1+|\lambda \eta|^2}} \varphi(\eta) d\eta, \end{aligned}$$

taking $t = 0$ derives

$$\mathcal{B}_0 f(x) = \lambda^n \int_{\mathbb{R}^n} e^{i\lambda x \cdot \eta} \varphi(\eta) d\eta = \lambda^n \hat{\varphi}(\lambda x).$$

Also, since

$$\hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(x) dx \geq \int_{\frac{5}{4} \leq |\xi| \leq \frac{7}{4}} \varphi(x) dx > 1,$$

there exists $0 < \delta < \frac{\lambda}{2}$ such that

$$\hat{\varphi}(\lambda x) > \frac{1}{2}, \quad \forall |x| < \frac{\delta}{\lambda}.$$

Accordingly,

$$|\mathcal{B}^{**} f(x)| \geq |\mathcal{B}_0 f(x)| \geq c_0 \lambda^n \quad \& \quad c_0 = \frac{1}{2(2\pi)^n}.$$

Now, if (1.13) holds, then

$$\begin{aligned} \lambda^{\frac{n}{2}+s} &\approx \|f\|_{\dot{H}^s(\mathbb{R}^n)} \\ &\gtrsim \left(\int_{\mathbb{R}^n} |\mathcal{B}^{**} f(x)|^q |x|^\alpha dx \right)^{\frac{1}{q}} \\ &\geq \left(\int_{B(0,1)} |\mathcal{B}^{**} f(x)|^q |x|^\alpha dx \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
&\geq c_0 \left(\int_{|x| < \frac{\delta}{\lambda}} \lambda^{nq} |x|^\alpha dx \right)^{\frac{1}{q}} \\
&= c_1 \lambda^{n - \frac{\alpha+n}{q}},
\end{aligned}$$

where

$$c_1 = c_0 \left(\frac{\omega_{n-1} \delta^{\alpha+n}}{\alpha+n} \right)^{\frac{1}{q}}$$

and ω_{n-1} is the area of unit sphere in \mathbb{R}^n . Therefore,

$$\lambda^{n - \frac{\alpha+n}{q}} \lesssim \lambda^{\frac{n}{2} + s}. \quad (4.9)$$

▷ Upon letting $\lambda \rightarrow \infty$ in (4.9), we get

$$\alpha \geq q \left(\frac{n}{2} - s \right) - n.$$

▷ Upon letting $\lambda \rightarrow 0$ in (4.9), we get

$$\alpha \leq q \left(\frac{n}{2} - s \right) - n.$$

As a consequence, we have

$$\alpha = q \left(\frac{n}{2} - s \right) - n. \quad \square$$

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